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CONTROL OF THE SPEED OF RESPONSE OF PREDATOR-PREY SYSTEMS*

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The problem of optimal of a predator-prey system is investigated. The existence of admissible control is established and the structure of optimal control is investigated.

Problems of optimal control of biological communities have been studied in many papers; bibliographies are contained in /1, 2/.

1. Statement of the problem. The dynamics of the interaction of predators and prey are described by the equation /3/

$$x_1^{\cdot}(\tau) = (a_1 - a_2 y_1) x_1, \; y_1^{\cdot}(\tau) = (a_3 x_1 - a_4) y_1 \tag{1.1}$$

where $x_1(\tau)$ is the population density of the prey and $y_1(\tau)$ that of the predators at time τ, and a_i are positive numbers characterizing the interspecific interactions.

In practice, to influence the system purposefully, one uses various chemical preparations such as pesticides, which act only on the prey, or only on the predators, or on both populations simultaneously.

First we will study the situation in which the control acts only on the prey. For the remaining two cases we restrict ourselves to describing the final result.

We will change to dimensionless variables given by the formulae

$$x_1(\tau) = a_4 a_3^{-1} x(t), \; y_1(\tau) = a_1 a_2^{-1} y(t), \; b = a_4 a_1^{-1}, \; \tau = a_1 t$$

Using the dimensionless variables in (1.1), the equations of the controlled system have the form

$$x^{\cdot}(t) = (1 - y) x - ux, \; y^{\cdot}(t) = b(x - 1) y \tag{1.2}$$

$$x(0) = x_0, \; y(0) = y_0, \; x_0 > 0, \; y_0 > 0, \; t \geqslant 0 \tag{1.3}$$

The control $u(t)$ satisfies the natural constraints

$$0 \leqslant u \leqslant \gamma, \; \gamma = \text{const} > 0 \tag{1.4}$$

For $u = 0$, system (1.2) has two equilibrium positions in the x, y plane: the points $(0, 0)$ and $(1, 1) = R$. Because only the point R is of any actual interest, the controllers' objective is to take system (1.2) from an arbitrary initial position (x_0, y_0) to the position R in the least possible time. Thus if $T(x_0, y_0, u)$ is the instant when system (1.2) first reaches the point R,

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the problem under consideration consists of the choice of a control function u_0 such that

$$\inf_u T(x_0, y_0, u) = T(x_0, y_0, u_0) \tag{1.5}$$

We remark that for $x_0 = 0$ (or $y_0 = 0$), (1.2) implies that $x(t) \equiv 0$ (or $y(t) \equiv 0$). Hence the problem should only be considered with the additional constraints $x_0 > 0$ and $y_0 > 0$.

Thus the statement of the problem comprises relations (1.2)-(1.5).

The solution of systems (1.2) and (1.3) for any control lies strictly inside the first quadrant G for all $t \geqslant 0$. In particular, for $u = 0$ the solutions of systems (1.2) and (1.3) form a nested family of closed curves (Fig.1), all containing the point R in their interiors /3/, i.e. to reach the equilibrium position the use of control measures is essential.

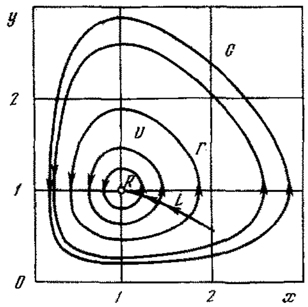

Fig.1

Note that the problem in question for $\gamma > 1$ with an integral criterion of benefit was considered in /2/.

2. ***Admissible control.*** By an admissible control we mean any measurable function $u(t)$ satisfying (1.4) for which there is a solution of (1.2) and (1.3) that reaches the point R in a finite time. We shall study the question of the existence of admissible controls.

We consider the time-reversed solution of system (1.2) which starts at the point R with control $u = \gamma$, i.e. a solution of the following problem:

$$x^{\cdot}(t) = (y-1)x + \gamma x,\ y^{\cdot}(t) = b(1-x)y \tag{2.1}$$
$$x(0) = 1,\ y(0) = 1,\ t \geqslant 0$$

On the basis of relations (2.1) there exists an instant of time t_1 such that the solution of problem (2.1) satisfies the inequalities $x(t) > 1$, $y(t) < 1$ for $t \in [0, t_1]$. We will denote this solution the interval $[0, t_1]$ by L.

We now consider the closed domain U, bounded on the outside by a trajectory of motion of system (1.2) with $u = 0$, which intersects L at some point (Fig.1). This trajectory, the boundary of the domain U, will be denoted by Γ. In the domain U we will set the control to be equal to zero everywhere outside the curve L, and equal to γ on L. By construction, for this choice of control the time taken for system (1.2) to reach the point R is finite for all $(x_0, y_0) \in U$, i.e. this control is admissible in the domain U.

We will now construct a control u under which the time taken to reach the domain U is finite for all $(x_0, y_0) \overline{\in} U$, $x_0 > 0$ and $y_0 > 0$. Let G_1 be the domain $\{0 < x \leqslant 1,\ y > 0\} \cup \{1 \leqslant x < 1 + \varepsilon,\ 0 < y < 1\}$, and let G_2 be the domain $\{G \setminus G_1\}$, where ε is a small number. In the domain $G_1 \setminus U$ the control $u = 0$, while in $G_2 \setminus U$ the control $u = \gamma$.

We will show that with this control u the time taken for system (1.2) to reach the domain U is finite for all (x_0, y_0). To do this we introduce the Lyapunov function

$$W = W(x(t), y(t)) = (x - 1 - \ln x) + b^{-1}(y - 1 - \ln y)$$

Note that the function W has been used in investigations of the stability of the Volterra model (see e.g. /4/).

We compute the total derivative of the function W along the trajectories of system (1.2) under control u:

$$W^{\cdot} = -u(x-1) \tag{2.2}$$

We now consider an arbitrary trajectory of system (1.2) under control u, starting at a point $(x_0, y_0) \overline{\in} U$. Suppose this trajectory does not reach the boundary Γ of the domain U in any finite time. In this case, because of the form of (2.2) and the positive-definiteness of the function W, the trajectory under consideration will always remain in a bounded domain, and consequently, from (1.2), it will circle the domain U an infinite number of times. During each circuit the time spent in the domain $G_2 \setminus U$ is bounded from below by some positive constant. But then, integrating both sides of (2.2) with respect to time, we find that $W(x(t), y(t)) \to -\infty$ for $t \to \infty$, which is impossible because it contradicts the positive-definiteness of the function W.

We have thus constructed an admissible control for the problem under consideration.

The existence of a fastest-acting control follows from this result and from /5/.

3. ***The structure of the optimal control (OC).*** We will investigate the form of the OC with the help of the maximum principle /6/. Let $\psi_1(t)$ and $\psi_2(t)$ be conjugate variables satisfying the equations

$$\psi_1^{\cdot}(t) = -\partial H/\partial x = \psi_1(y - 1 + u) - b\psi_2 y \tag{3.1}$$
$$\psi_2^{\cdot}(t) = -\partial H/\partial y = \psi_1 x + b\psi_2(1 - x)$$
$$H = \psi_0 + \psi_1(1 - y - u) + b\psi_2(x - 1)y,\ \psi_0 = \text{const} \leqslant 0 \tag{3.2}$$

On the basis of the existence of an OC established above, and the maximum principle, we will find a non-zero solution of Eqs.(3.1) such that the function H reaches its maximum value, qual to zero, at the OC. By (3.2) this means that the optimal control u_0 is found by maximising the expression $\varphi = -\psi_1 ux$ with respect to u. Hence it is clear that $u_0 = \gamma$ if $\psi_1 < 0$ and $u_0 = 0$ if $\psi_1 > 0$.

We will show that the function ψ_1 does not vanish over whole intervals. Suppose this is not the case and that the function $\psi_1(t)$ vanishes in some interval $I \subset [0, T]$. Then $\psi_1^{\cdot}(t) = 0$ for $t \in I$. Consequently, from the first equation of (3.1), we also have $\psi_2(t) = 0$ for $t \in I$. Hence the conjugate variables ψ_1 and ψ_2 simultaneously vanish, which is impossible by the maximum principle. It can be similarly shown that the zeros of the function $\psi_1(t)$ are simple. Consequently, the optimal control u_0 takes only its extreme values (i.e. either 0 or γ) and is a piecewise constant function.

Further construction of the OC of system (1.2), together with its optimal trajectories, is performed numerically. There are two cases: $\gamma \geqslant 1$ and $0 < \gamma < 1$.

The results of computer calculations of optimal trajectories (OT) and "switch" lines (SL) for system (1.2) with parameters $b = 1$ and $\gamma = 1$ are shown in Fig.2. The lines AR and RB are SLs, the curve AR being a controlled trajectory of system (1.2). The procedure for constructing the OTs and the SL RB was based on comparing the times taken by system (1.2) to reach the equilibrium position R from various initial states (x_0, y_0) with various control laws. The points of the curve RB together with the optimal control law for system (1.2) were found from this comparison. If at the initial moment $t_0 = 0$ the system is in position Q, then it moves along a controlled trajectory $(u = \gamma)$ until it reaches position S. At the point S the control switches and the system begins to move along the uncontrolled trajectory $(u = 0)$ to the point P, where there is another control switch and the system moves along the SL AR to position R.

If the initial position of the system is the point N, then the system moves along the uncontrolled trajectory $(u = 0)$ to the point P. There a switch occurs and the system moves along the controlled trajectory to the point R.

The results obtained show that for $\gamma \geqslant 1$ and from any initial position (x_0, y_0), system (1.2) reaches the equilibrium position R with no more than two switches in the optimal control.

Fig.3 shows the x-dependence of the Bellman function $V(x, y)$ (the value of the least time) for the case under consideration.

It is clear that as x increases, the time of the system to reach the equilibrium position increases if $y > 1$, but that if $y \leqslant 1$ the value of $V(x, y)$ decreases as the initial position approaches the SL AR from the left. As soon as the initial position is to the right of the curve AR, there is a sharp increase in the least time.

When γ is increased $(\gamma \geqslant 1)$, the SL shown in Fig.2 remain essentially the same. The point S approaches R, the curvature of the curve SR decreases and the points of the curve APR shift to the right.

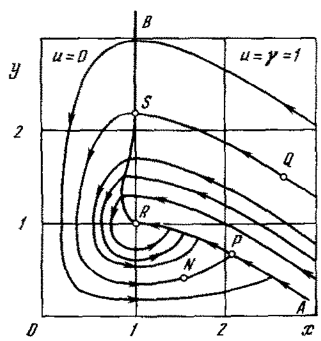

Fig.2

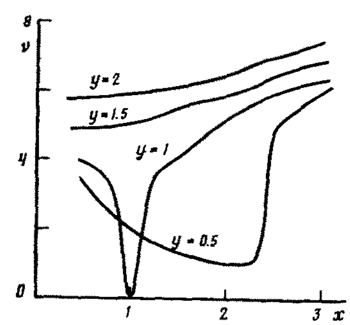

Fig.3

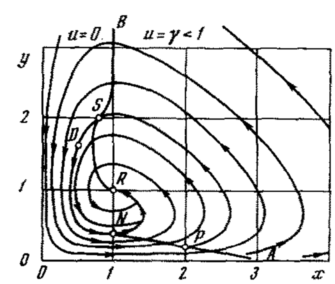

Fig.4

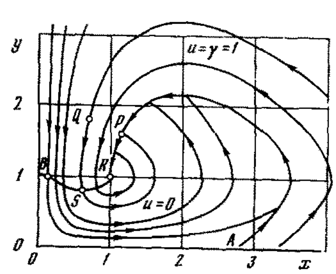

Fig.5

Fig.4 shows the OTs and the SL $ANRSB$ of system (1.2) for $b=1$ and $\gamma=0,4$. It is clear that here the situation is quite different from the case $\gamma \geqslant 1$. The section NR of the switch line $ANRSB$ is a segment of a trajectory of system (1.2) with $u=\gamma$. The point R can be reached with no more than two switches of the OC only if the initial position lies in the closed domain bounded by the curve $PSDNP$. If the initial position lies outside this domain, then the number of switch points on the OT will always be greater than two. For $x_0<1$ and $y_0<1$, the point (x_0, y_0) approaches R and the time taken to reach R and the number of switches of the OT decreases as the values of the initial coordinates increase. If however $x_0>1$ and $y_0>1$, then as they increase the number of switch points of the OC and the time to reach the equilibrium position R increase.

4. We will give the results of computer calculations for the case when the control (pesticides) acts only on the predators, i.e. for a controlled system of the form

$$x^{\cdot}(t)=(1-y)x,\ y^{\cdot}(t)=b(x-1)y-uy \tag{4.1}$$

$$x(0)=x_0,\ y(0)=y_0,\ x_0>0,\ y_0>0,\ t\geqslant 0 \tag{4.2}$$

where the control u satisfies the constraints (1.4).

In this case the situation is qualitatively independent of the value of γ. Fig.5 shows the OTs and the SL $APRSB$ of system (4.1) for $b=1$ and $\gamma=1$. The segment APR is part of a trajectory of system (4.1) for $u=\gamma$. The curves in Fig.5 show that the OC of system (4.1) has no more than two switch points.

The least time was also computed. The calculations showed that when y increases from values smaller than unity to values greater than unity with x fixed, there is at first a decrease in the time taken to reach the equilibrium position, and then a sharp increase. Similar results were obtained for fixed y and for x increasing from values smaller than unity to values greater than unity.

Changing γ does not significantly affect the form of the controlled trajectories of system (4.1); as γ increases their concavity increases and the point A shifts to the right. The number of switches of an OT of system (4.1) is never greater than two.

The speed-of-response problem was similarly investigated for arbitrary values of the parameters b and γ.

We note that the problem of the control of the speed of response of a system of the form (4.1), in which the term u replaced the term uy, (i.e. the effectiveness of the pesticide does not depend on the number of predators), was investigated in /1/.

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